Introduction to Fair Division

Computational Social Choice

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Outline

- What is *Social Choice*?
  - Making a $\underbrace{\textit{collective decision}}_{\textit{society}}$ based on the $\underbrace{\textit{individual preferences}}_{\textit{choices}}$.

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  - Maximize individual happiness (Fair)
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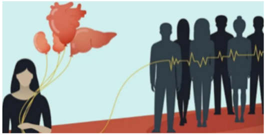
- Do such ‘good’ decisions always exist?
- What *Computation* has to do with it?
  - If such solutions exist, can they be computed efficiently?

Divisible resources

Problem Setting: Allocate Resources!

Indivisible resources



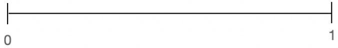
Fair Division of Divisible Items

Cake Division Model

An Allocation $X = (X_1, X_2)$

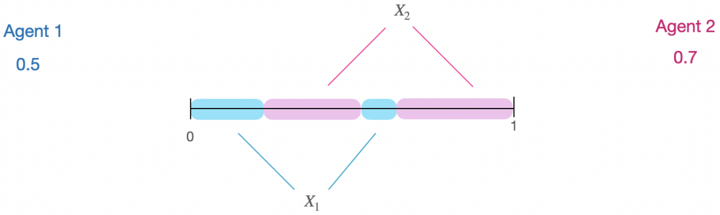
Agent 1

Agent 2



- Valuations functions: $v_i(X) = 1$

Cake Division Model



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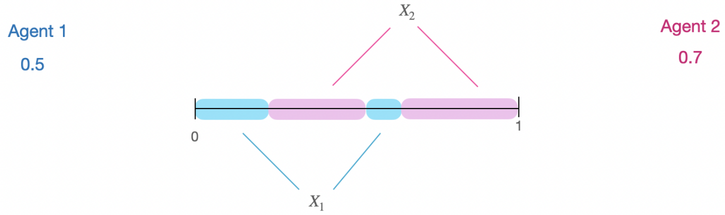

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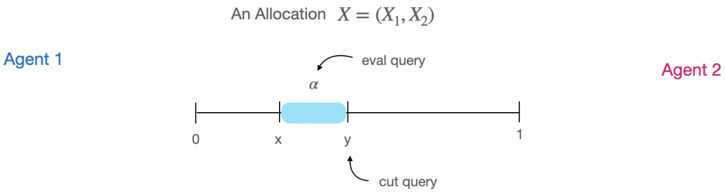
- Robertson-Web Model:
  - $eval_i(x, y) = v_i(x, y)$
  - $cut_i(x, \alpha) = y$ such that $v_i(x, y) = \alpha$

Cake Division Model



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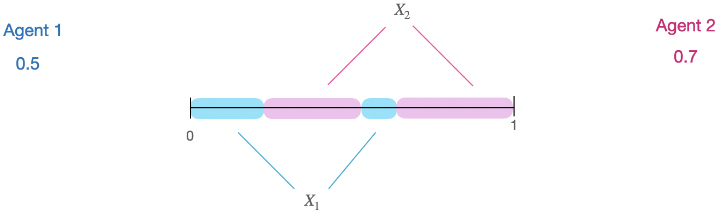
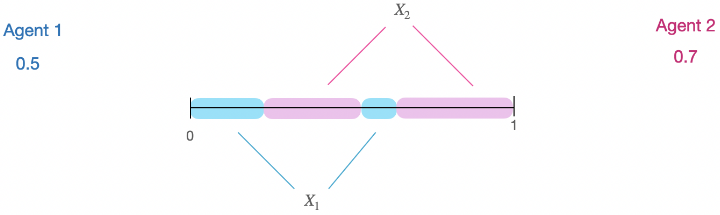


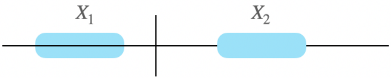
- ([Steinhaus, 1948](#)) Proportionality: $v_i(X_i) \geq \frac{1}{n}$



- (Steinhaus, 1948) Proportionality: $v_i(X_i) \geq \frac{1}{n}$
- (Gamow and Stern, 1958; Foley, 1967) Envy-Freeness: $v_i(X_i) \geq v_i(X_j)$

Envy-Free Cake Division for 2 Agents

Agent 1:



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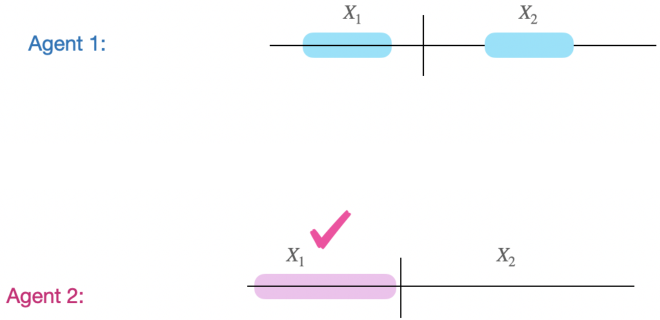

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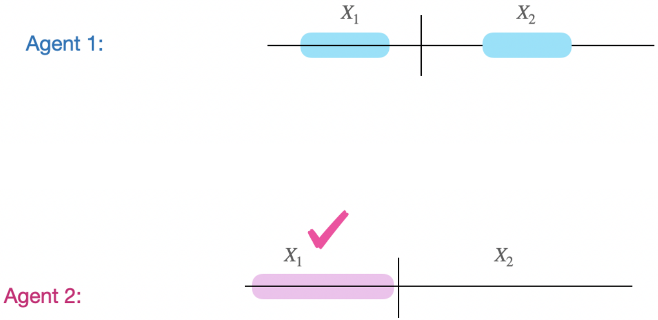

- Envy-Free
- Also Proportional: $v_1(X_1) = \frac{1}{2}$ and $v_2(X_2) \geq \frac{1}{2}$

Proportional Cake Division for n Agents

Dubins-Spanier Procedure

$$v_i(X_i) \geq \frac{1}{n}$$

- A knife moves on the interval $[0, 1]$
- An agent i shouts when the knife reaches a point y such that $v_i([0, y]) = \frac{1}{n}$
- The agent leaves with the piece $[0, y]$
- The process repeats with the remaining agents

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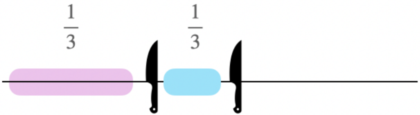

Envy-Free Cake Division for 3 Agents

Selfridge-Conway Algorithm (1960's)

- Agents 1, 2, 3 and a Cake $C = [0, 1]$
- Valuations v_1, v_2, v_3

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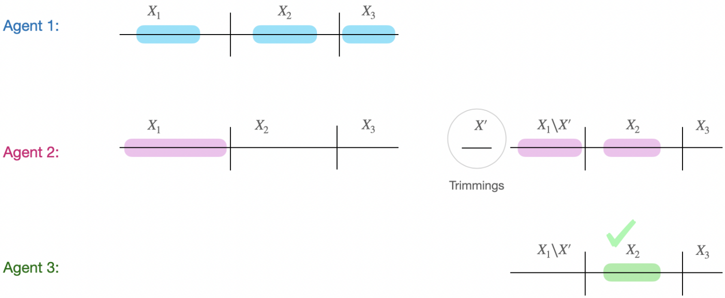

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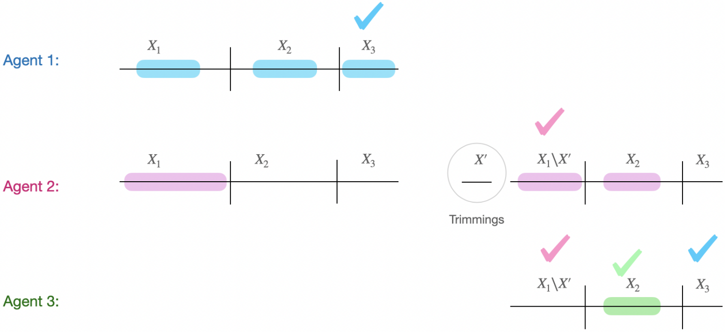

Envy-Free Cake Division for 3 Agents

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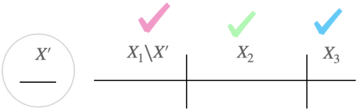
The Trimmed Piece



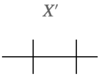
Trimmings

Agent 1:

Agent 2:

Agent 3:

1,



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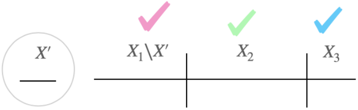
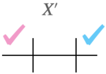
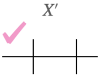


Trimmings

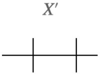
Agent 1: 3.



Agent 2: 2.

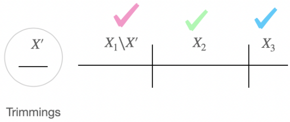


Agent 3: 1,



Gets the last left piece

Selfridge-Conway Algorithm (1960's)
The Trimmed Piece



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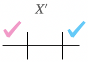

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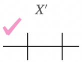

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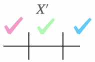

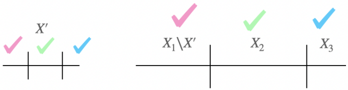

Envy-Free Cake Division: The Story

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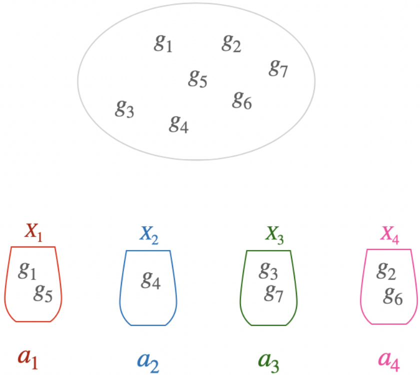
- (Proccacia, 2009) Any EF protocol requires at least $\Omega(n^2)$ queries.

Fair Division of Indivisible Items

Dividing the Indivisible!

- Set of Items $\{g_1, g_2, \ldots g_m\}$
- Set of Agents $\{a_1, a_2, \ldots a_n\}$



Dividing the Indivisible!

	g_1	g_2	g_3	g_4	g_5
a_1	5	10	2	3	10
a_2	10	5	2	4	12

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$$v_1(X_1) = v_1(\{g_1, g_2\}) = v_1(g_1) + v_1(g_2) = 5 + 10 = 15$$

Dividing the Indivisible!

EF does not exist for indivisible items!

Deciding if there is an EF allocation is NP-Hard even for binary valuations!

Dividing the Indivisible

- (Budish 2011) EF upto *one* item (EF1): $v_i(X_i) \geq v_i(X_j \setminus \{g\})$

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	g_1	g_2	g_3
a_1	$\boxed{1}$	1	2
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EF1 but not EFX

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EF1 but not EFX

EF $\Rightarrow$ EFX $\Rightarrow$ EF1

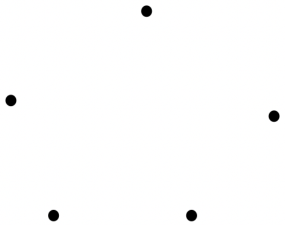
Results At a Glance

- EF1 always exists (even for monotone valuations)
- EFX always exists for 2 agents (Plaut and Roughgarden 2020), 3 agents (Chaudhury et al. 2020), 2 types of agents (Mahara 2023)

Do EFX Allocations always exists?

EF1 Allocations

Envy Cycle Elimination (Lipton et al. 2004)



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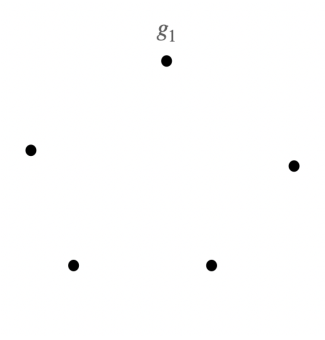

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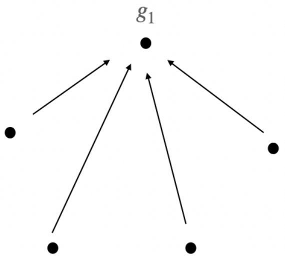

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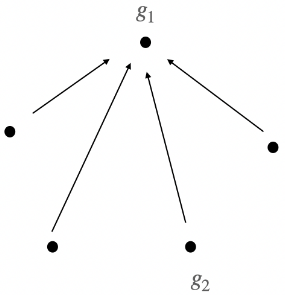

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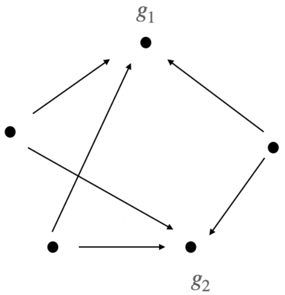

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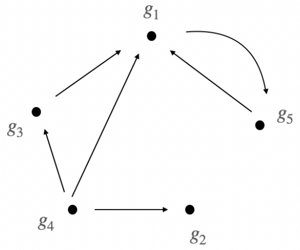

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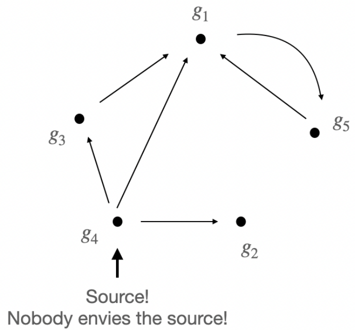

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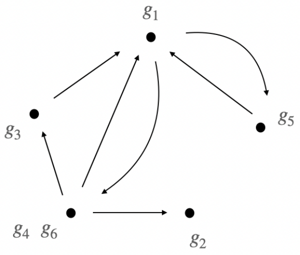

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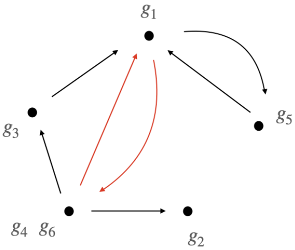
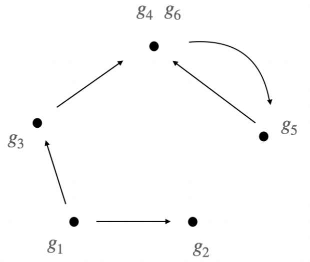


No Source!
But there is an Envy Cycle!

Envy Cycle Elimination



EF1 Allocations

Envy Cycle Elimination
While there is a good g to be allocated:

- Construct Envy Graph of the partial allocation A
- Find a source in the Envy Graph and allocate g to the source
- If there is no source, then eliminate the envy cycles by rotating the bundles on the cycle

The process terminated in polynomial time
The final allocation is EF1

Thank you!